Combined Universal Generating Function and Semi-Markov Process Technique for Multi-state System Reliability Evaluation

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Abstract

The paper presents an approach for multi-state system reliability evaluation where times to failures and times to repair are non-exponentially distributed. The suggested procedure reduces a problem dimension and simplifies solution.

1. Introduction

Traditional binary reliability models allow only two functional states for a system and for each of its components: perfect functionality (UP) and complete failure (DOWN). Multi-state System (MSS) reliability analysis relates to systems for which one cannot formulate an "all or nothing" type of failure criterion. Many real-world systems are composed of multi-state components, which have different performance levels and several failure modes with various effects on the entire system performance. Reliability models were extended from binary state (UP or DOWN) to finite number of performance levels (Multi-State System models). A semi-Markov processes method [Limnios and Oprisan (2001)] is a powerful approach and its application to reliability evaluation gives the opportunity to get solution for a MSS where failure and repair times may be arbitrary (non-exponentially) distributed. The main difficulty of semi-Markov processes application to reliability evaluation for complex MSS is the "dimension damnation". At first, state-space diagram building of semi-Markov model for MSS is not a simple job. It is a difficult non-formalized process that may cause numerous mistakes even for relatively small MSS. At second, to get solution for high-order systems of equations (especially integral equations for semi-Markov model) or simulation of complex MSS with large number of system states can require enormous computational efforts. Indeed, the number of integral equations in the system that should be solved using semi-Markov approach is equal to the square of total number of MSS states. For MSS consisting of n different repairable elements where every element j has k_j different performance levels one will have MSS with $K = \prod_{j=1}^{n} k_j$ states. Therefore, the total number of integral equations (that should be solved in order to find states probabilities for the MSS by using straightforward semi-Markov method) will be K^2. This number can be very large even for relatively small MSS.

Therefore, the development of the method based on simplified procedures which can reduce the problem dimension may be extremely beneficial for reliability engineers. The paper presents such a method that uses a special mathematical technique – Universal Generating Function (UGF) and called as *combined UGF and semi-Markov processes method*. UGF technique was introduced by Ushakov (1986). More details about the UGF one can find in [Gnedenko and Ushakov (1995)]. The combined UGF and random processes method was primary developed only for Markov processes by Lisnianski and Levitin (2003). In the presented paper the method was extended to semi-Markov processes where times to failure and times to repair may be non-exponentially (arbitrary) distributed.

2. Model Description

In general case any element j in MSS can have k_j different states corresponding to different performance, represented by the set $\boldsymbol{g}_j=\{g_{j1},...,g_{jk_j}\}$, where g_{ji} is the performance rate of element j in the state i, $i \in \{1, 2, ..., k_j\}$. The generic MSS model [Natvig (1984)] consists of the performance stochastic processes $G_j(t)$, $G_j(t) \in \boldsymbol{g}_j$, $j=1, \ldots, n$ for each system element j, and the system structure function that produces the stochastic process corresponding to the output performance of the entire MSS: $G(t) = \phi(G_1(t),...,G_n(t))$.

2.1. Semi-Markov Model for Multi-state Element

According to the method at first stage a semi-Markov model for every multi-state element should be built. By this way a state probabilities $p_{ji}(t) = \Pr\{G_j(t) = g_{ji}\}$, $i \in \{1, ..., k_j\}$ for every MSS's element $j \in \{1, ..., n\}$ can be obtained.

We consider a multi-state element with minor failure and repairs as it was defined by Lisnianski and Levitin (2003). With every state i there is associated performance g_{ji} of the element j. The states are ordered so that $g_{ji+1} \geq g_{ji}$ for any i. Minor failures and repairs cause element transitions from state i, where $1 \leq i \leq k_j$, only to the adjacent states. It will be transition to the state i-1 if failure occurs in the state i and it will be transition to the state $i+1$ after repair. In the state k_j it may be only the failure and transition to the state k_j-1 and in the state 1 it may be only the repair and transition to the state 2.

For every element j, $1 \leq j \leq n$ we assume that time to failure is distributed according to cumulative distribution function (c.d.f.) $F_{i,i-1}^{(j)}(t)$ for any state i, $1 < i \leq k_j$. Analogously, for any state i, $1 \leq i < k_j$ time to repair assumed to be distributed according to c.d.f. $F_{i,i+1}^{(j)}(t)$.

In order to define semi-Markov process, we obtain the kernel matrix $\boldsymbol{Q}_j(t)=\left|Q_{lm}^{(j)}(t)\right|$, $l,m=1,2,\ldots k_j$:

$$\boldsymbol{Q}_j(t)=\begin{vmatrix} 0 & Q_{12}^{(j)}(t) & 0 & 0 & \ldots & 0 & 0 \\ Q_{21}^{(J)}(t) & 0 & Q_{23}^{(j)}(t) & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & Q_{k_j k_j-1}^{(j)}(t) & 0 \end{vmatrix} \quad (1)$$

in which

$$Q_{12}^{(j)}(t) = F_{1,2}^{(j)}(t), \quad (2)$$

$$Q_{l+1,l}^{(j)}(t) = \int_0^t [1 - F_{l+1,l+2}^{(j)}(t)] dF_{l+1,l}^{(j)}(t) \quad \text{for } 1 \leq l \leq k_j - 2, \quad (3)$$

$$Q_{l,l+1}^{(j)}(t) = \int_0^t [1 - F_{l,l-1}^{(j)}(t)] dF_{l,l+1}^{(j)}(t) \quad \text{for } 2 \leq l \leq k_j - 1, \quad (4)$$

$$Q^{(j)}_{k_j\,k_j-1}(t) = F^{(j)}_{k_j,k_j-1}(t)\,. \qquad (5)$$

Each element $Q^{(j)}_{lm}(t)$ of this matrix determines the probability that transition from state l to state m occurs during time interval [0, t]. The kernel matrix (1) and the initial state k_j (with the best performance) completely define the semi-Markov process which describes the stochastic behaviour of any multi-state element j.

For every element j we designate $\theta^{(j)}_{lm}(t)$ the probability that semi-Markov stochastic process which starts from initial state l at instant t=0, will be in state m at instant t. Probabilities $\theta^{(j)}_{lm}(t)$, l,m=1, 2, …, k_j can be found from the solution of the following system of integral equations:

$$\theta^{(j)}_{lm}(t) = \delta_{lm}[1-\sum_{m=1}^{k_j} Q^{(j)}_{lm}(t)] + \sum_{s=1}^{k_j}\int_0^t q^{(j)}_{ls}(\tau)\theta^{(j)}_{sm}(t-\tau)d\tau\,, \qquad (6)$$

where

$$q^{(j)}_{is}(\tau) = \frac{dQ^{(j)}_{is}(\tau)}{d\tau} \quad \text{and} \quad \delta_{lm}=1,\ \text{if}\ l=m,\ \delta_{lm}=0,\ \text{if}\ l \neq m\,.$$

In our case the process always starts from the state k_j, hence the states probabilities of multi-state element j which should be defined from the system of integral equations (6) are the following

$$p_{j\,k_j}(t) = \theta^{(j)}_{k_j\,k_j}(t)\,,\ \ p_{j\,k_j-1}(t) = \theta^{(j)}_{k_j\,k_j-1}(t)\,,\ \ldots\ ,\ p_{j\,2}(t) = \theta^{(j)}_{k_j\,2}(t)\,,\ p_{j\,1}(t) = \theta^{(j)}_{k_j\,1}(t)\,. \quad (7)$$

So, at the first stage "small" semi-Markov models should be built for each element of the entire MSS. In general case for any element j the semi-Markov model consists of $lm = k_j^2$ integral equations (6). By solving these equations, we obtain the performance probability distribution (7) $p_{ji}(t) = \Pr\{G_j(t) = g_{ji}\}$, i=1, …, k_j for every element $j = 1$, …, n at each time instant t.

2.2. Multi-state System Reliability Evaluation

At the second stage based on determined states probabilities for all elements, UGF for each individual element should be defined. Then by using composition operators over UGF of individual elements and their combinations in the entire MSS structure, one can obtain the resulting UGF for the entire MSS by using simple algebraic operations. This UGF defines the output performance distribution for the entire MSS at each time instant t. MSS reliability measures can be easily derived from this output performance distribution.

The following steps should be executed at the second stage.

1. Having the sets $\boldsymbol{g}_j$ and probabilities $p_{ji}(t)$ for each element j define universal generating function (UGF) of this element in the form

$$u_j(z)=p_{j1}(t)z^{g_{j1}}+p_{j2}(t)z^{g_{j2}}+\ldots+p_{jk_j}(t)z^{g_{jk_j}}\ . \qquad (8)$$

2. Using the composition operators $\Omega_{\phi s}$ and $\Omega_{\phi p}$ defined by Lisnianski and Levitin (2003), over the UGF of individual elements and their combinations and applying the recursive procedure for series-parallel systems or using the operators described for the bridge structures, obtain the resulting UGF for the entire MSS:

$$U(z,t) = \sum_{i=1}^{K} p_i(t) z^{g_i}\ , \qquad (9)$$

where K is the number of entire system states and g_i is the entire system performance in the corresponding state i.

3. Applying the operators δ_A, δ_E, δ_D introduced in [Lisnianski and Levitin (2003)] over the resulting UGF of the entire MSS one can obtain the following MSS reliability indices:

a. MSS availability $A(t, w)$ at instant t>0 for arbitrary constant demand w

$$A(t,w) = \delta_A(U(z,t),w) = \delta_A(\sum_{i=1}^{K} p_i(t) z^{g_i}, w) = \sum_{i=1}^{K} p_i(t) 1(F(g_i, w) \geq 0), \qquad (10)$$

where $F(g_i, w)=g_i-w$ is an acceptability function.

b. MSS expected output performance at instant t>0

$$E(t) = \delta_E(U(z,t)) = \delta_E(\sum_{i=1}^{K} p_i(t) z^{g_i}) = \sum_{i=1}^{K} p_i(t) g_i. \qquad (11)$$

c. MSS expected performance deficiency at t>0 for arbitrary constant demand w

$$D(t,w) = \delta_D(U(z),w) = \delta_D(\sum_{i=1}^{K} p_i(t) z^{g_i}, w) = \sum_{i=1}^{K} p_i(t) \cdot \max(w - g_i, 0). \qquad (12)$$

3. Conclusions

The advantages of the proposed approach are:

* Simplification of semi-Markov model building. Instead of the building of complex semi-Markov model for the entire MSS, one should built n separate relatively simple semi-Markov models for system elements.
* Simplification of the process of solving a system of equations. Instead of solving one high-order system of $(\prod_{j=1}^{n} k_j)^2$ integral equations one has to solve n low-order systems. In each system the number of integral equations is lower or equal than $\max_{j \in \{1,\dots n\}} \{k_j^2\}$.

Numerical examples prove the approach efficiency.

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